

SIMPLICIAL RANDOM VARIABLES

IVAN MARIN

ABSTRACT. We introduce a new ‘geometric realization’ of an (abstract) simplicial complex, inspired by probability theory. This space (and its completion) is a metric space, which has the right (weak) homotopy type, and which can be compared with the usual geometric realization through a natural map, which has probabilistic meaning : it associates to a random variable its probability mass function. This ‘probability map’ function is proved to be a (Serre) fibration and a (weak) homotopy equivalence.

CONTENTS

1. Introduction and main results	1
2. Simplicial properties and completion	2
2.1. Functorial properties	3
2.2. Technical preliminaries	3
2.3. Weak homotopy equivalence	4
3. Homotopies inside $L(\Omega, \{0, 1\})$	5
4. Probability law	8
4.1. The law maps	8
4.2. Homotopy lifting properties	9
References	12

1. INTRODUCTION AND MAIN RESULTS

In this paper we consider a new ‘geometric realization’ of an (abstract) simplicial complex, inspired by probability theory. This space is a metric space, which has the right (weak) homotopy type, and can be compared with the usual geometric realization through map, which is very natural in probabilistic terms : it associated to a random variable its probability mass function. This ‘probability map’ function is proved to be a (Serre) fibration and a (weak) homotopy equivalence. This construction passes to the completion, and has nice functorial property.

We specify the details now. Let S be a set, and $\mathcal{P}_f(S)$ the set of its finite subsets. We set $\mathcal{P}_f^*(S) = \mathcal{P}_f(S) \setminus \{\emptyset\}$. Recall that an (abstract) simplicial complex is a collection of subsets $\mathcal{K} \subset \mathcal{P}_f^*(S)$ with the property that, for all $X \in \mathcal{K}$ and $Y \in \mathcal{P}_f^*(S)$, $Y \subset X \Rightarrow Y \in \mathcal{K}$. The elements of \mathcal{K} are called its faces, and the vertices of \mathcal{K} are the union of the elements of \mathcal{K} .

We endow S with the discrete metric of diameter 1, and with the Borel σ -algebra associated to this topology. We define $L(\Omega, S)$ as the set of random variables/measurable maps $\Omega \rightarrow S$ up to neglectability, endowed with the metric $d(f, g) = \int_{\Omega} d(f(t), g(t)) dt$. We define $L(\Omega, \mathcal{K})$ as the subset of $L(\Omega, S)$ made of the (equivalence classes of) measurable maps $f : \Omega \rightarrow S$ such that $\{s \in S \mid \lambda(f^{-1}(\{s\})) > 0\} \in \mathcal{K}$.

Recall that the (usual) ‘geometric’ realization of \mathcal{K} is defined as

$$|\mathcal{K}| = \{t : S \rightarrow [0, 1] \mid \{s \in S; t_s > 0\} \in \mathcal{K} \text{ \& \; } \sum_{s \in S} t_s = 1\}$$

and that its topology is given by the direct limit of the $[0, 1]^A$ for $A \in \mathcal{P}_f(S)$. There is a natural map $L(\Omega, \mathcal{K}) \rightarrow |\mathcal{K}|$ which associates to $f : \Omega \rightarrow \mathcal{K}$ the element $t : S \rightarrow [0, 1]$ defined by $t_s = \lambda(f^{-1}(\{s\}))$. In probabilistic terms, it associates to the random variable f its probability law, or probability

mass function. We denote $|\mathcal{K}|_1$ the same set as $|\mathcal{K}|$, but with the topology defined by the metric $|\alpha - \beta|_1 = \sum_{s \in S} |\alpha(s) - \beta(s)|$. We denote $\overline{|\mathcal{K}|_1}$ its completion as a metric space.

It is easily checked that, unless S is finite, $L(\Omega, \mathcal{K})$ is not in general closed in $L(\Omega, S)$, and therefore not complete. We denote $\bar{L}(\Omega, \mathcal{K})$ its closure inside $L(\Omega, S)$. The ‘probability law’ map $\Psi : L(\Omega, \mathcal{K}) \rightarrow |\mathcal{K}|_1$ is actually continuous, and can be extended to a map $\bar{\Psi} : \bar{L}(\Omega, \mathcal{K}) \rightarrow \overline{|\mathcal{K}|_1}$. Keane’s theorem about the contractibility of $\text{Aut}(\Omega)$ (see [2]) easily implies that these maps have contractible fibers. The goal of this note is to specify the homotopy-theoretic features of them. We get the following results.

Theorem 1.1.

- (1) *The map $L(\Omega, \mathcal{K}) \rightarrow \bar{L}(\Omega, \mathcal{K})$ is a weak homotopy equivalence.*
- (2) *The ‘probability law’ map $L(\Omega, \mathcal{K}) \rightarrow |\mathcal{K}|_1$ is a Serre fibration and a weak homotopy equivalence. It admits a continuous global section.*
- (3) *The ‘probability law’ map $\bar{L}(\Omega, \mathcal{K}) \rightarrow \overline{|\mathcal{K}|_1}$ is a Serre fibration and a weak homotopy equivalence. It admits a continuous global section.*
- (4) *$L(\Omega, \mathcal{K})$ and $\bar{L}(\Omega, \mathcal{K})$ have the same weak homotopy type as the ‘geometric realization’ $|\mathcal{K}|$ of \mathcal{K} .*

In other terms, in the commutative diagram below, the vertical maps are Serre fibrations, and all the maps involved are weak homotopy equivalences, the map $|\mathcal{K}| \rightarrow |\mathcal{K}|_1$ being in addition a strong homotopy equivalence. Actually, when \mathcal{K} is finite, we prove that all the maps are strong homotopy equivalences (see theorem 4.6).

$$\begin{array}{ccc} L(\Omega, \mathcal{K}) & \hookrightarrow & \bar{L}(\Omega, \mathcal{K}) \\ \Psi_{\mathcal{K}} \downarrow & & \downarrow \bar{\Psi}_{\mathcal{K}} \\ |\mathcal{K}| & \longrightarrow & |\mathcal{K}|_1 \hookrightarrow \overline{|\mathcal{K}|_1} \end{array}$$

We now comment on the functorial properties of this construction. By definition, a morphism $\varphi : \mathcal{K}_1 \rightarrow \mathcal{K}_2$ between simplicial complexes is a map from the set $\bigcup \mathcal{K}_1$ of vertices of \mathcal{K}_1 to the set of vertices of \mathcal{K}_2 with the property that $\forall F \in \mathcal{K}_1 \ \varphi(F) \in \mathcal{K}_2$. We denote **Simp** the corresponding category of simplicial complexes.

Let **Set** denote the category of sets and **Met**₁ denote the full subcategory of the category metric spaces and contracting maps made of the spaces of diameter at most 1. Let **CMet**₁ be the full subcategory of **Met**₁ made of complete metric spaces. There is a completion functor $\text{Comp} : \mathbf{Met}_1 \rightarrow \mathbf{CMet}_1$ which associates to each metric space its completion. Then $L(\Omega, \bullet) : X \rightsquigarrow L(\Omega, X)$ defines a functor **Set** \rightarrow **CMet**₁ (see [3]). It can be decomposed as $L(\Omega, \bullet) = \text{Comp} \circ L_f(\Omega, \bullet)$ where $L_f(\Omega, S)$ is the subspace of $L(\Omega, S)$ made of the (equivalence classes of) functions $f : \Omega \rightarrow S$ of essentially finite image, that is such that there exists $S_0 \subset S$ finite such that $\sum_{s \in S_0} \lambda(f^{-1}(\{s\})) = 1$.

We prove in section 2.1 below that our simplicial constructions have similar functorial properties, which can be summed up as follows.

Proposition 1.2. *$L(\Omega, \bullet)$ and $\bar{L}(\Omega, \bullet)$ define functors **Simp** \rightarrow **Met**₁ and **Simp** \rightarrow **CMet**₁, with the property that $\bar{L}(\Omega, \bullet) = \text{Comp} \circ L(\Omega, \bullet)$.*

Acknowledgements. I thank D. Chataur and A. Rivière for discussions and references.

2. SIMPLICIAL PROPERTIES AND COMPLETION

In this section we prove part (1) of theorem 1.1. We start by proving the functorial properties stated in the introduction.

2.1. Functorial properties. We denote, as in the previous section, $\bar{L}(\Omega, \mathcal{K})$ the closure of $L(\Omega, \mathcal{K})$ inside $L(\Omega, S)$. As a closed subset of a complete metric space, it is a complete metric space. For any $f \in L(\Omega, S)$, we denote

$$f(\Omega) = \{s \in S \mid \lambda(f^{-1}(\{s\})) > 0\}$$

the essential image of an arbitrary measurable map $\Omega \rightarrow S$ representing f .

Lemma 2.1. *Let $f \in L(\Omega, S)$ and $S_f = \{s \in S \mid \lambda(f^{-1}(\{s\})) > 0\}$. Then $f \in \bar{L}(\Omega, \mathcal{K})$ if and only if $\mathcal{P}_f(S_f) \setminus \{\emptyset\} \subset \mathcal{K}$.*

Proof. Let $f \in \bar{L}(\Omega, \mathcal{K})$, and $S_f = f(\Omega) = \{s \in S \mid \lambda(f^{-1}(\{s\})) > 0\}$ as in the statement. Let $F \subset S_f$ be a finite subset. We set $m = \min\{\lambda(f^{-1}(\{s\})) \mid s \in F\}$. We have $m > 0$. Since $f \in \bar{L}(\Omega, \mathcal{K})$, there exists $f_0 \in L(\Omega, \mathcal{K})$ such that $d(f, f_0) < m$. We then have $F \subset f_0(\Omega)$. Indeed, there would otherwise exist $s \in F \setminus f_0(\Omega)$, and then $d(f, f_0) \geq \lambda(f^{-1}(\{s\})) \geq m$, a contradiction. From this we get that $\mathcal{P}_f(S_f) \setminus \{\emptyset\} \subset \mathcal{K}$. Conversely, let $f \in L(\Omega, S)$ such that $\mathcal{P}_f(S_f) \setminus \{\emptyset\} \subset \mathcal{K}$. From [3] proposition 3.3 we know that $S_f = f(\Omega) \subset S$ is countable. If S_f is finite we have $S_f \in \mathcal{P}_f(S_f) \setminus \{\emptyset\}$ hence $S_f \in \mathcal{K}$ and $f \in L(\Omega, \mathcal{K})$. Otherwise, let us fix a bijection $\mathbf{N} \rightarrow S_f$, $n \mapsto x_n$ and define $f_n \in L(\Omega, S)$ by $f_n(t) = f(t)$ if $f(t) \in \{x_0, \dots, x_n\}$, and $f_n(t) = x_0$ otherwise. Clearly $f_n(\Omega) \in \mathcal{P}_f(S_f) \setminus \{\emptyset\} \subset \mathcal{K}$, hence $f_n \in L(\Omega, \mathcal{K})$. On the other hand, $d(f_n, f) \leq \sum_{k>n} \lambda(f^{-1}(\{x_k\})) \rightarrow 0$, hence $f \in \bar{L}(\Omega, \mathcal{K})$. \square

We prove that, as announced in the introduction, $L(\Omega, \bullet)$ provides a functor $\mathbf{Simp} \rightarrow \mathbf{CMet}_1$ that can be decomposed as $Comp \circ L_f(\Omega, \bullet)$. If $f \in L(\Omega, \mathcal{K}_1)$, $g = L(\Omega, \varphi)(f) = \varphi \circ f$ is a measurable map and $g(\Omega) = \varphi(f(\Omega))$. Since $f(\Omega) \in \mathcal{K}_1$ and φ is simplicial we get that $\varphi(f(\Omega)) \in \mathcal{K}_2$ hence $g \in L(\Omega, \mathcal{K}_2)$. From this one gets immediately that $L(\Omega, \bullet)$ defines a functor $\mathbf{Simp} \rightarrow \mathbf{Met}_1$.

Similarly, if $f \in \bar{L}(\Omega, \mathcal{K}_1)$ and $g = \varphi \circ f = L(\Omega, \varphi)(f) \in L(\Omega, S)$, then again $g(\Omega) = \varphi(f(\Omega))$. But, for any finite set $F \subset g(\Omega) = \varphi(f(\Omega))$ there exists $F' \subset f(\Omega)$ finite and with the property that $F' = \varphi(F)$. Now $f \in \bar{L}(\Omega, \mathcal{K}_1) \Rightarrow F' \in \mathcal{K}_1$, by lemma 2.1, hence $F \in \mathcal{K}_2$ because φ is a simplicial morphism. By lemma 2.1 one gets $g \in \bar{L}(\Omega, \mathcal{K}_2)$, hence $\bar{L}(\Omega, \bullet)$ defines a functor $\mathbf{Simp} \rightarrow \mathbf{CMet}_1$. We check immediately that $\bar{L}(\Omega, \bullet) = Comp \circ L(\Omega, \bullet)$, and this proves proposition 1.2.

2.2. Technical preliminaries.

Lemma 2.2. *Let F be a set. The map $c \mapsto \{t; c(t) \notin F\}$ is uniformly continuous $L(\Omega, S) \rightarrow L(\Omega, 2)$, and even contracting.*

Proof. Let $f_1, f_2 \in L(\Omega, S)$, and $\Psi : L(\Omega, S) \rightarrow L(\Omega, 2)$ the map defined by the statement. Then $\Psi(f_1)(t) \neq \Psi(f_2)(t) \Rightarrow f_1(t) \neq f_2(t)$, hence $d(\Psi(f_1)(t), \Psi(f_2)(t)) \leq d(f_1(t), f_2(t))$ for all $t \in \Omega$ and finally $d(\Psi(f_1), \Psi(f_2)) \leq d(f_1, f_2)$, whence Ψ is contracting and uniformly continuous. \square

Lemma 2.3. *Let $a, b, c, d \in \mathbf{R}$ with $a \leq b$ and $c \leq d$. Then*

$$\lambda([a, b] \setminus [c, d]) \leq \max(|a - c|, |b - d|) \leq |a - c| + |b - d|.$$

Proof. Straightforward. \square

Lemma 2.4. *Let $\Delta_r = \{\underline{\alpha} = (\alpha_1, \dots, \alpha_r) \in \mathbf{R}_+^r \mid \alpha_1 + \dots + \alpha_r = 1\}$ denote the r -dimensional simplex. The map $\Delta_r \rightarrow L(\Omega, \{1, \dots, r\})$ defined by $\underline{\alpha} \mapsto f_{\underline{\alpha}}$ where $f_{\underline{\alpha}}(t) = i$ iff $t \in [\alpha_1 + \dots + \alpha_{i-1}, \alpha_1 + \dots + \alpha_i]$ is continuous. More precisely it is $2r$ -Lipschitz if Δ_r is equipped with the metric $d(\underline{\alpha}, \underline{\alpha}') = \sum_i |\alpha_i - \alpha'_i|$.*

Proof. Let $\underline{\alpha}, \underline{\alpha}' \in \Delta_r$. We denote $\beta_i = \alpha_1 + \dots + \alpha_i$, $\beta_0 = 0$, and we similarly define the β'_i . We have $\beta_i - \beta_{i-1} = \alpha_i$ hence $|\beta'_i - \beta_i| \leq \sum_{k \leq i} |\alpha'_k - \alpha_k|$ and finally $\sum_i |\beta'_i - \beta_i| \leq r \sum_i |\alpha'_i - \alpha_i|$. Now, for $t \in [\beta_i, \beta_{i+1}]$ we have $f_{\underline{\alpha}}(t) = f_{\underline{\alpha}'}(t)$ unless $t \notin [\beta'_i, \beta'_{i+1}]$. From this and lemma 2.3 we get

$$d(f_{\underline{\alpha}}, f_{\underline{\alpha}'}) \leq \sum_{i=1}^r \lambda([\beta_i, \beta_{i+1}] \setminus [\beta'_i, \beta'_{i+1}]) \leq \sum_{i=1}^r |\beta_i - \beta'_i| + |\beta_{i+1} - \beta'_{i+1}| \leq 2 \sum_{i=1}^r |\beta_i - \beta'_i| \leq 2r \sum_{i=1}^r |\alpha_i - \alpha'_i|$$

and this proves the claim. \square

Lemma 2.5. *Let \mathcal{K} be a simplicial complex and X a topological space, and $A \subset X$. If $\gamma_0, \gamma_1 : X \rightarrow \bar{L}(\Omega, \mathcal{K})$ are two continuous maps such that $\forall x \in X \quad \gamma_0(x)(\Omega) \subset \gamma_1(x)(\Omega)$, and $(\gamma_0)|_A = (\gamma_1)|_A$, then γ_0 and γ_1 are homotopic relatively to A . Moreover, if γ_0 and γ_1 take value inside $L(\Omega, \mathcal{K})$, then the homotopy takes values inside $L(\Omega, \mathcal{K})$.*

Proof. We fix an identification $\Omega \simeq [0, 1]$. We define $H : [0, 1] \times X \rightarrow L(\Omega, S)$ by $H(u, x)(t) = \gamma_0(x)(t)$ if $t \geq u$ and $H(u, x)(t) = \gamma_1(x)(t)$ if $t < u$. We have $H(0, \bullet) = \gamma_0$ and $H(1, \bullet) = \gamma_1$. On the other hand, for all $u \in [0, 1]$ and $x \in \Omega$ we have $H(u, x)(\Omega) \subset \gamma_0(x)(\Omega) \cup \gamma_1(x)(\Omega) = \gamma_1(x)(\Omega)$. Therefore $H(u, x) \in \mathcal{K}$ si $\gamma_1 \in L(\Omega, \mathcal{K})$, and $\mathcal{P}_f(H(u, x)(\Omega)) \subset \mathcal{P}_f(\gamma_1(x)(\Omega)) \subset \mathcal{K}$ if $\gamma_1 \in \bar{L}(\Omega, \mathcal{K})$. From this we get that H takes values inside $\bar{L}(\Omega, \mathcal{K})$, and even inside $L(\Omega, \mathcal{K})$ if $\gamma_1 \in L(\Omega, \mathcal{K})$. Moreover, it is clear that $\gamma_0(x) = \gamma_1(x)$ implies $H(u, x) = \gamma_0(x) = \gamma_1(x)$ for all $u \in [0, 1]$, therefore the homotopy indeed fixes A . Finally, we need to check that H is continuous over $[0, 1] \times X$, and this is a consequence of $d(H(x, u), H(x, v)) \leq |u - v|$ for all $u, v \in [0, 1]$. \square

2.3. Weak homotopy equivalence. We now proves part (1) of the main theorem, through a series of propositions, which might be of independent interest.

Proposition 2.6. *Let C be a compact subspace of $\bar{L}(\Omega, \mathcal{K})$ and $C_0 \subset C \cap L(\Omega, \mathcal{K})$ a compact subspace (possibly empty) such that $\bigcup_{c \in C_0} c(\Omega)$ is finite. Then there exists a continuous map $p : C \rightarrow L(\Omega, \mathcal{K})$ such that $p(c) = c$ for all $c \in C_0$. Moreover, $p(c)(\Omega) \subset c(\Omega)$ for all $c \in C$ and $\bigcup_{c \in C} p(c)(\Omega)$ is finite.*

Proof. For any $F \in \mathcal{P}_f(S)^*$ and $n \in \mathbb{N}^*$, we denote $O_{F,n} = \{f \in L(\Omega, S) \mid \forall s \in F \quad \lambda(f^{-1}(\{s\})) > 1/n\}$. It is an open subset of $L(\Omega, S)$. Let $\mathcal{E} = \bigcup_{c \in C} \mathcal{P}_f(c(\Omega))^*$. By lemma 2.1, the open subsets $O_{F,n}$ for $F \in \mathcal{E}$ cover $\bar{L}(\Omega, \mathcal{K})$, hence because C is compact there exists $F_1, \dots, F_r \in \mathcal{E}$ and $n_1, \dots, n_r \in \mathbb{N}^*$ such that $\bar{L}(\Omega, \mathcal{K}) \subset \bigcup_{i=1}^r O_{F_i, n_i}$. Up to replacing them by their maximum, we may suppose $n_1 = \dots = n_r = n_0$. Let then $F' = \bigcup_{c \in C_0} c(\Omega) \in \mathcal{E}$. We set $F = F_1 \cup F_2 \cup \dots \cup F_r \cup F'$. It is a finite set. For any $i \in \{1, \dots, r\}$ we set $O_i = O_{F_i, n_0}$ and we choose $x_i \in F_i$.

For any $c \in C$, we set

$$\alpha_i(c) = \frac{d(c, {}^c O_i)}{\sum_j d(c, {}^c O_j)}$$

and $\beta_i(c) = \sum_{k \leq i} \alpha_k(c)$. These define continuous maps $C \rightarrow \mathbf{R}_+$. We then set

$$\begin{aligned} p(c)(t) &= c(t) && \text{si } c(t) \in F, \text{ i.e. } t \notin \Omega_c \\ &= x_i && \text{si } t \in \Omega_c \cap [\beta_{i-1}, \beta_i[\end{aligned}$$

Let $c_1, c_2 \in C$ et $\underline{\alpha}^s$, $s = 1, 2$ the corresponding r -tuples $\underline{\alpha}^s = (\alpha_1^s, \dots, \alpha_r^s)$. We have

$$d(p(c_1), p(c_2)) = \int_{\Omega_{c_1} \cup \Omega_{c_2}} d(p(c_1)(t), p(c_2)(t)) dt \leq \lambda(\Omega_{c_1} \Delta \Omega_{c_2}) + \int_{\Omega_{c_1} \cap \Omega_{c_2}} d(p(c_1)(t), p(c_2)(t)) dt$$

and we know $\lambda(\Omega_{c_1} \Delta \Omega_{c_2}) \leq d(c_1, c_2)$ because of lemma 2.2. Therefore we only need to check that the term $\int_{\Omega_{c_1} \cap \Omega_{c_2}} d(p(c_1)(t), p(c_2)(t)) dt$ is continuous. But

$$\int_{\Omega_{c_1} \cap \Omega_{c_2}} d(p(c_1)(t), p(c_2)(t)) dt = \int_{\Omega_{c_1} \cap \Omega_{c_2}} d(f_{\underline{\alpha}^1}(t), f_{\underline{\alpha}^2}(t)) dt \leq d(f_{\underline{\alpha}^1}, f_{\underline{\alpha}^2}) \leq 2r|\underline{\alpha}^1 - \underline{\alpha}^2|$$

by lemma 2.4, whence the conclusion, by continuity of $c \mapsto \underline{\alpha}$.

We must now check that p takes values inside $L(\Omega, \mathcal{K})$. Let $c \in C$. If $c(\Omega) \subset F$ this is clear. Otherwise, $p(c)(\Omega)$ is finite, and $p(c)(\Omega) \subset c(\Omega) \cup \{x_i; c \in O_i\}$. But $c \in O_i$ implies $x_i \in F_i \subset c(\Omega)$, hence $p(c)(\Omega) \subset c(\Omega)$ and $p(c)(\Omega) \in \mathcal{P}_f^*(c(\Omega)) \subset \mathcal{K}$ by lemma 2.1, since $c \in \bar{L}(\Omega, \mathcal{K})$. Finally, we have $p(c) = c$ for all $c \in C_0$ since $F \supset F'$. \square

We immediately get the following corollary, by letting $C_0 = \{c_1^0, \dots, c_k^0\}$.

Corollary 2.7. *Let C be a compact subset of $\bar{L}(\Omega, \mathcal{K})$ and $c_1^0, \dots, c_k^0 \in C \cap L(\Omega, \mathcal{K})$. Then there exists a continuous map $p : C \rightarrow L(\Omega, \mathcal{K})$ such that $p(c_i^0) = c_i^0$ for all $i \in \{1, \dots, k\}$. Moreover, $p(c)(\Omega) \subset c(\Omega)$ for all $c \in C$ and $\bigcup_{c \in C} p(c)(\Omega)$ is finite.*

Proposition 2.8. *Let C be a compact space, and $x_0 \in C$. For any simplicial complex \mathcal{K} , and any continuous map $\gamma : C \rightarrow L(\Omega, \mathcal{K})$, there exists a continuous map $\hat{\gamma} : (C, x_0) \rightarrow (L(\Omega, \mathcal{K}), \gamma(x_0))$ which is homotopic to γ relatively to $(\{x_0\}, \{\gamma(x_0)\})$, and such that $\bigcup_{x \in S^n} \hat{\gamma}(x)(\Omega)$ is finite.*

Proof. Let $C' = \gamma(C) \subset L(\Omega, \mathcal{K})$. It is compact, hence applying corollary 2.7 to it and to $\{c_1^0\} = \{\gamma(x_0)\}$ we get a continuous map $p : C' \rightarrow L(\Omega, \mathcal{K})$ such that $\bigcup_{c \in C'} p(c)(\Omega)$ is finite, and $p(c)(\Omega) \subset c(\Omega)$ for all $c \in C'$. Therefore, letting $\hat{\gamma} = p \circ \gamma : C \rightarrow L(\Omega, \mathcal{K})$, we get that $\bigcup_{x \in C} \hat{\gamma}(x)(\Omega)$ is finite. Since $\hat{\gamma}(x)(\Omega) \subset \gamma(x)(\Omega)$ for all $x \in C$, we get from lemma 2.5 that γ and $\hat{\gamma}$ are homotopic, hence the conclusion. \square

Proposition 2.9. *Let C be a compact space (and $x_0 \in C$) a simplicial complex \mathcal{K} , and a pair of continuous maps $\gamma_0, \gamma_1 : C \rightarrow L(\Omega, \mathcal{K})$ (with $\gamma_0(x_0) = \gamma_1(x_0)$). If γ_0 and γ_1 are homotopic as maps in $\bar{L}(\Omega, \mathcal{K})$ (relatively to $(\{x_0\}, \{\gamma_0(x_0)\})$), then they are homotopic inside $L(\Omega, \mathcal{K})$ (relatively to $(\{x_0\}, \{\gamma_0(x_0)\})$).*

Proof. After proposition 2.8, there exists $\hat{\gamma}_0, \hat{\gamma}_1 : C \rightarrow L(\Omega, \mathcal{K})$ such that $\hat{\gamma}_i$ is homotopic to γ_i with the property that $\bigcup_{x \in C} \hat{\gamma}_i(x)(\Omega)$ is finite, for all $i \in \{0, 1\}$. Without loss of generality, one can therefore assume that $\bigcup_{x \in |\Delta_n|} \gamma_i(x)(\Omega)$ is finite, for all $i \in \{0, 1\}$. Let $H : C \times [0, 1] \rightarrow \bar{L}(\Omega, \mathcal{K})$ be an homotopy between γ_0 and γ_1 . Let $C' = H(C \times [0, 1])$ and $C_0 = \gamma_0(C) \cup \gamma_1(C)$. These are two compact spaces which satisfy the assumptions of proposition 2.6. If $p : C \rightarrow L(\Omega, \mathcal{K})$ is the continuous map afforded by this proposition, then $\hat{H} = p \circ H$ provides an homotopy between γ_0 and γ_1 inside $L(\Omega, \mathcal{K})$. The ‘relative’ version of the statement is proved similarly. \square

In particular, when C is equal to the n -sphere S^n , this proves that the natural map $[S^n, L(\Omega, \mathcal{K})]_* \rightarrow [S^n, \bar{L}(\Omega, \mathcal{K})]_*$ between sets of pointed homotopy classes is injective. In order to prove theorem 1.1 (1), we need to prove that it is surjective. Let us consider a continuous map $\gamma : S^n \rightarrow \bar{L}(\Omega, \mathcal{K})$ and set $C = \gamma(S^n)$. It is a compact subspace of $\bar{L}(\Omega, \mathcal{K})$. Applying proposition 2.6 with $C_0 = \emptyset$ we get $p : C \rightarrow L(\Omega, \mathcal{K})$ such that $p(c)(\Omega) \subset c(\Omega)$ for any $c \in C$. Let then $\hat{\gamma} = p \circ \gamma : S^n \rightarrow L(\Omega, \mathcal{K})$. From lemma 2.5 we deduce that $\hat{\gamma}$ and γ are homotopic inside $\bar{L}(\Omega, \mathcal{K})$, and this concludes the proof of part (1) of theorem 1.1.

3. HOMOTOPIES INSIDE $L(\Omega, \{0, 1\})$

In this section we denote $L(2) = L(\Omega, 2) = L(\Omega, \{0, 1\})$, with $d(0, 1) = 1$. Since we are going to use Lipschitz properties of maps, we specify our conventions on metrics. When (X, d_X) and (Y, d_Y) are two metric spaces, we endow $X \times Y$ with the metric $d_X + d_Y$, and $C^0(I, X)$ with the metric of uniform convergence $d(\alpha, \beta) = \|\alpha - \beta\|_\infty = \sup_{t \in I} |\alpha(t) - \beta(t)|$. Recall that the topology on $C^0(I, X)$ induced by this metric is the compact-open topology.

Identifying $L(2) = L(\Omega, 2)$ with the space of Borel subsets of Ω with the metric $d(E, F) = \lambda(E \Delta F)$, we have the following lemma. This lemma can be viewed as providing a continuous reparametrization by arc-length of natural geodesics inside the metric space $L(2)$.

Lemma 3.1. *The exists a continuous map $\mathbf{g} : L(2) \times [0, 1] \rightarrow L(2)$ such that $\mathbf{g}(A, 0) = A$, $\lambda(\mathbf{g}(A, u)) = \lambda(A)(1 - u)$ and $\mathbf{g}(A, u) \supset \mathbf{g}(A, v)$ for all A and $u \leq v$. Moreover, it satisfies*

$$\mathbf{g}(E, u) \Delta \mathbf{g}(F, v) \leq 4\lambda(E \Delta F) + |v - u|$$

for all $E, F \in L(2)$ and $u, v \in [0, 1]$.

Proof. For $E \in L(2) \setminus \{\emptyset\}$ we define $\varphi_E(t) = \lambda(E \cap [t, 1]) / \lambda(E)$. The map φ_E is obviously (weakly) decreasing and continuous $[0, 1] \rightarrow [0, 1]$, with $\varphi_E(0) = 1$ and $\varphi_E(1) = 0$. It is therefore surjective, and we can define a (weakly) decreasing map $\psi_E : [0, 1] \rightarrow [0, 1]$ by $\psi_E(u) = \inf \varphi_E^{-1}(\{u\})$. Since φ_E is continuous, we have $\varphi_E(\psi_E(u)) = u$.

One defines $\mathbf{g}(E, u) = E \cap [\psi_E(1 - u), 1]$ if $\lambda(E) \neq 0$, and $\mathbf{g}(\emptyset, u) = \emptyset$. We have $\lambda(\mathbf{g}(E, u)) = \lambda(E \cap [\psi_E(1 - u), 1]) = \varphi_E(\psi_E(1 - u))\lambda(E) = (1 - u)\lambda(E)$ when $\lambda(E) \neq 0$, and $\lambda(\mathbf{g}(\emptyset, u)) = 0 = \lambda(E)(1 - u)$ if $\lambda(E) = 0$. It is clear that $\mathbf{g}(E, u) \subset \mathbf{g}(E, v)$ for all $u \geq v$.

Moreover, clearly $\mathbf{g}(E, 0) = E$ since $E \cap [\psi_E(1), 1] \subset E$ and $\lambda(E \cap [\psi_E(1), 1]) = \varphi_E(\psi_E(1))\lambda(E) = \lambda(E)$. It remains to prove that \mathbf{g} is continuous.

Let $E, F \in L(2)$ and $u, v \in [0, 1]$. We first assume $\lambda(E)\lambda(F) > 0$. Without loss of generality we can assume $\psi_E(1 - u) \leq \psi_F(1 - v)$. Then $[\psi_E(1 - u), 1] \supset [\psi_F(1 - v), 1]$, and $\mathbf{g}(E, u)\Delta\mathbf{g}(F, v)$ can be decomposed as

$$((E \setminus F) \cap [\psi_E(1 - u), 1]) \cup ((F \setminus E) \cap [\psi_F(1 - v), 1]) \cup ((E \cap F) \cap [\psi_E(1 - u), \psi_F(1 - v)]).$$

Since the first two pieces are included inside $E\Delta F$, we get $\lambda(\mathbf{g}(E, u)\Delta\mathbf{g}(F, v)) \leq \lambda(E\Delta F) + \lambda((E \cap F) \cap [\psi_E(1 - u), \psi_F(1 - v)])$. Now $(E \cap F) \cap [\psi_E(1 - u), \psi_F(1 - v)] = (E \cap F \cap [\psi_E(1 - u), 1]) \setminus (E \cap F \cap [\psi_F(1 - v), 1])$ hence

$$\begin{aligned} \lambda((E \cap F) \cap [\psi_E(1 - u), \psi_F(1 - v)]) &= \lambda(E \cap F \cap [\psi_E(1 - u), 1]) - \lambda(E \cap F \cap [\psi_F(1 - v), 1]) \\ &\leq \lambda(E \cap [\psi_E(1 - u), 1]) - \lambda(E \cap F \cap [\psi_F(1 - v), 1]) \\ &\leq (1 - u)\lambda(E) - \lambda(E \cap F \cap [\psi_F(1 - v), 1]) \end{aligned}$$

Now, since $F = (E \cap F) \sqcup (F \setminus E)$, we have $F \cap [\psi_F(1 - v), 1] = ((E \cap F) \cap [\psi_F(1 - v), 1]) \sqcup ((F \setminus E) \cap [\psi_F(1 - v), 1])$ hence

$$\begin{aligned} (1 - v)\lambda(F) &= \lambda((E \cap F) \cap [\psi_F(1 - v), 1]) + \lambda((F \setminus E) \cap [\psi_F(1 - v), 1]) \\ &\leq \lambda((E \cap F) \cap [\psi_F(1 - v), 1]) + \lambda(F \setminus E) \\ &\leq \lambda((E \cap F) \cap [\psi_F(1 - v), 1]) + \lambda(F\Delta E). \end{aligned}$$

It follows that $-\lambda((E \cap F) \cap [\psi_F(1 - v), 1]) \leq \lambda(F\Delta E) - (1 - v)\lambda(F)$ hence

$$\lambda((E \cap F) \cap [\psi_E(1 - u), \psi_F(1 - v)]) \leq (1 - u)\lambda(E) + \lambda(F\Delta E) - (1 - v)\lambda(F)$$

and finally

$$\begin{aligned} \mathbf{g}(E, u)\Delta\mathbf{g}(F, v) &\leq 2\lambda(E\Delta F) + (1 - u)\lambda(E) - (1 - v)\lambda(F) \\ &\leq 2\lambda(E\Delta F) + (\lambda(E) - \lambda(F)) + (v - u)\lambda(E) + v(\lambda(F) - \lambda(E)) \\ &\leq 2\lambda(E\Delta F) + |\lambda(E) - \lambda(F)| + |v - u|\lambda(E) + v|\lambda(F) - \lambda(E)| \\ &\leq 2\lambda(E\Delta F) + 2|\lambda(E) - \lambda(F)| + |v - u| \\ &\leq 4\lambda(E\Delta F) + |v - u|. \end{aligned}$$

Therefore we get the inequality $\mathbf{g}(E, u)\Delta\mathbf{g}(F, v) \leq 4\lambda(E\Delta F) + |v - u|$, that we readily check to hold also when $\lambda(E)\lambda(F) = 0$. This proves that \mathbf{g} is continuous, whence the claim. \square

This lemma is actually all what is needed to prove theorem 1.1 in the case of binary random variables, that is $S = \{0, 1\}$, as we illustrate below (see corollary 4.5). In the general case however, we shall need a more powerful homotopy, provided by proposition 3.6 below. The next lemmas are preliminary technical steps in view of its proof.

Lemma 3.2. *The map $C^0(I, L(2)) \times L(2) \rightarrow C^0(I, L(2))$ defined by $(E_\bullet, A) \rightarrow \alpha$ where $\alpha(u) = \lambda(E_u \cap A)$, is 1-Lipschitz.*

Proof. Let α, β denote the images of (E_\bullet, A) and (F_\bullet, B) , respectively. Then, for all $u \in I$, we have

$$|\alpha(u) - \beta(u)| = |\lambda(E_u \cap A) - \lambda(F_u \cap B)| \leq \lambda((E_u \cap A)\Delta(F_u \cap B))$$

From the general set-theoretic inequality $(X \cap A)\Delta(Y \cap B) \subset (X\Delta Y) \cup (A\Delta B)$ one gets

$$\lambda((E_u \cap A)\Delta(F_u \cap B)) \leq \lambda(E_u\Delta F_u) + \lambda(A\Delta B),$$

hence $|\alpha - \beta|_\infty \leq \sup_u \lambda(E_u\Delta F_u) + \lambda(A\Delta B)$ and this proves the claim. \square

Lemma 3.3. *A map $\Phi_- : C^0(I, [0, 1]) \times C^0(I, L(2)) \times L(2) \rightarrow C^0(I, L(2))$ is defined as follows. To $(a, E_\bullet, A) \in C^0(I, [0, 1]) \times C^0(I, L(2)) \times L(2) \rightarrow C^0(I, L(2))$ one associates the map*

$$\Phi_-(a, E_\bullet, A) : u \mapsto \mathbf{g}\left(E_u \cap A, 1 - \frac{\min(a(u)\lambda(E_u), \alpha(u))}{\alpha(u)}\right)$$

if $\alpha(u) \neq 0$, and otherwise $u \mapsto \emptyset$, where $\alpha(u) = \lambda(A \cap E_u)$. Then, the map Φ_- is continuous.

Proof. Let us fix $(a, E_\bullet, A) \in C^0(I, [0, 1]) \times C^0(I, L(2)) \times L(2)$, and let $\varepsilon > 0$. Consider $\hat{m} : [0, 1] \times [\varepsilon/12, 1] \rightarrow [0, 1]$ be defined by $\hat{m}(x, y) = \min(x, y)/y$. It is clearly continuous on the compact space $[0, 1] \times [\varepsilon/12, 1]$, hence uniformly continuous, hence there exists $\eta > 0$ such that $\max(|x_1 - x_2|, |y_1 - y_2|) < \eta \Rightarrow |\hat{m}(x_1, y_1) - \hat{m}(x_2, y_2)| \leq \varepsilon/6$. Clearly one can assume $\eta \leq \varepsilon/6$ as well.

Let us then consider $(b, F_\bullet, B) \in C^0(I, [0, 1]) \times C^0(I, L(2)) \times L(2)$ such that $\|a - b\|_\infty + \sup_u \lambda(E_u \Delta F_u) + \lambda(A \Delta B) \leq \eta$. We set $\alpha(u) = \lambda(E_u \cap A)/\lambda(E_u)$ and $\beta(u) = \lambda(F_u \cap B)/\lambda(F_u)$ with $\alpha(u) = 0$ if $\lambda(E_u) = 0$, $\beta(u) = 0$ if $\lambda(F_u) = 0$. From lemma 3.2, we have $\|\alpha - \beta\|_\infty \leq \eta$. Let us consider $I_0 = \{u \in I \mid \alpha(u) \leq \varepsilon/3\}$. We have by definition $\alpha(I \setminus I_0) \subset]\varepsilon/3, 1] \subset [\varepsilon/12, 1]$ and, since $\|\alpha - \beta\|_\infty \leq \varepsilon/6$, we have $\beta(I \setminus I_0) \subset]\varepsilon/6, 1] \subset [\varepsilon/12, 1]$. Moreover, since $|a(u)\lambda(E_u) - b(u)\lambda(F_u)| \leq |a(u) - b(u)|\lambda(E_u) + b(u)|\lambda(E_u) - \lambda(F_u)| \leq |a(u) - b(u)| + \lambda(E_u \Delta F_u) \leq \eta$, we get that, for all $u \notin I_0$, we have $|\hat{m}(a(u)\lambda(E_u), \alpha(u)) - \hat{m}(b(u)\lambda(F_u), \beta(u))| \leq \varepsilon/6$. Moreover, since in particular $\alpha(u)\beta(u) \neq 0$, we get from the general inequality $\mathbf{g}(X, x)\Delta\mathbf{g}(Y, y) \leq 4\lambda(X\Delta Y) + |x - y|$ of lemma 3.1 that, for all $u \notin I_0$,

$$\begin{aligned} d(\Phi_-(a, E_\bullet, A)(u), \Phi_-(b, F_\bullet, B)(u)) &\leq 4\lambda((E_u \cap A)\Delta(F_u \cap B)) \\ &\quad + |\hat{m}(a(u)\lambda(E_u), \alpha(u)) - \hat{m}(b(u)\lambda(F_u), \beta(u))| \\ &\leq 4(\lambda(E_u \Delta F_u) + \lambda(A \Delta B)) + \varepsilon/6 \\ &\leq 4\varepsilon/6 + \varepsilon/6 \\ &< \varepsilon \end{aligned}$$

Now, if $u \in I_0$, then $\Phi_-(a, E_\bullet, A)(u) \subset E_u \cap A$ hence $\lambda(\Phi_-(a, E_\bullet, A)(u)) \leq \lambda(E_u \cap A) = \alpha(u) \leq \varepsilon/3$ and $\lambda(\Phi_-(b, F_\bullet, B)(u)) \leq \lambda(F_u \cap B) = \beta(u) \leq \varepsilon/3 + \varepsilon/6 = \varepsilon/2$, whence

$$d(\Phi_-(a, E_\bullet, A)(u), \Phi_-(b, F_\bullet, B)(u)) \leq \lambda(\Phi_-(a, E_\bullet, A)(u)) + \lambda(\Phi_-(b, F_\bullet, B)(u)) \leq 5\varepsilon/6 < \varepsilon.$$

It follows that $d(\Phi_-(a, E_\bullet, A), \Phi_-(b, F_\bullet, B)) \leq \varepsilon$ and Φ_- is continuous at (a, E_\bullet, A) , which proves the claim. \square

We use the convention $\mathbf{g}(X, t) = X$ for $t \leq 0$ and $\mathbf{g}(X, t) = \emptyset$ for $t > 1$, so that \mathbf{g} is extended to a continuous map $L(2) \times \mathbf{R} \rightarrow L(2)$.

Lemma 3.4. *A map $\Phi_+ : C^0(I, [0, 1]) \times C^0(I, L(2)) \times L(2) \rightarrow C^0(I, L(2))$ is defined as follows. To $(a, E_\bullet, A) \in C^0(I, [0, 1]) \times C^0(I, L(2)) \times L(2) \rightarrow C^0(I, L(2))$ one associates the map*

$$\Phi_+(a, E_\bullet, A) : u \mapsto \mathbf{g}\left(E_u \cap^c A, 1 - \frac{\max(0, a(u)\lambda(E_u) - \alpha(u))}{\lambda(E_u) - \alpha(u)}\right)$$

if $\alpha(u) \neq \lambda(E_u)$, and otherwise $u \mapsto \emptyset$, where $\alpha(u) = \lambda(A \cap E_u)$. Then, the map Φ_+ is continuous.

The proof is similar to the one of the previous proposition, and left to the reader.

Lemma 3.5. *The map $(f, g) \mapsto (t \mapsto f(t) \cup g(t))$ is continuous $C^0(I, L(2))^2 \rightarrow C^0(I, L(2))$, and even 1-Lipschitz.*

Proof. The map $(X, Y) \mapsto X \cup Y$ is 1-Lipchitz because of the general set-theoretic fact $(X_1 \cup Y_1)\Delta(X_2 \cup Y_2) \subset (X_1 \Delta X_2) \cup (Y_1 \Delta Y_2)$ from which we deduce $\lambda((X_1 \cup Y_1)\Delta(X_2 \cup Y_2)) \leq \lambda(X_1 \Delta X_2) + \lambda(Y_1 \Delta Y_2)$, which proves that $(X, Y) \mapsto X \cup Y$ is 1-Lipschitz $L(2)^2 \rightarrow L(2)$. It follows that the induced map $C^0(I, L(2))^2 = C^0(I, L(2))^2 \rightarrow C^0(I, L(2))$ is 1-Lipschitz and thus continuous, too. \square

Proposition 3.6. *There exists a continuous map $\Phi : C^0(I, [0, 1]) \times C^0(I, L(2)) \times L(2) \rightarrow C^0(I, L(2))$ with the property that, for all $(a, E_\bullet, A) \in C^0(I, [0, 1]) \times C^0(I, L(2)) \times L(2)$ such that $A \subset E_0$ and $a(0)\lambda(E_0) = \lambda(A)$, we have $\Phi(a, E_\bullet, A)(0) = A$ and, for all $u \in I$, $\Phi(a, E_\bullet, A)(u) \subset E_u$ and $\lambda(\Phi(a, E_\bullet, A)(u)) = a(u)\lambda(E_u)$.*

Proof. We define $\Phi(a, E_\bullet, A)(u) = \Phi_-(a, E_\bullet, A)(u) \cup \Phi_+(a, E_\bullet, A)(u)$. By combining lemmas 3.3, 3.4 and 3.5 we get that Φ is continuous. Moreover, $\Phi_-(a, E_\bullet, A)(u) \subset E_u \cap A$ and $\Phi_+(a, E_\bullet, A)(u) \subset E_u \cap^c A$ hence $\Phi(a, E_\bullet, A)(u) = \Phi_-(a, E_\bullet, A)(u) \sqcup \Phi_+(a, E_\bullet, A)(u) \subset E_u$, with $\lambda(\Phi(a, E_\bullet, A)(u)) =$

$\lambda(\Phi_-(a, E_\bullet, A)(u)) + \lambda(\Phi_+(a, E_\bullet, A)(u))$. Letting $\alpha(u) = \lambda(E_u \cap A)$, again by lemmas 3.3 and 3.4 we get

$$\lambda(\Phi_-(a, E_\bullet, A)(u)) = \lambda\left(\mathbf{g}\left(E_u \cap A, 1 - \frac{\min(a(u)\lambda(E_u), \alpha(u))}{\alpha(u)}\right)\right) = \min(a(u)\lambda(E_u), \alpha(u))$$

and

$$\lambda(\Phi_+(a, E_\bullet, A)(u)) = \lambda\left(\mathbf{g}\left(E_u \cap^c A, 1 - \frac{\max(0, a(u)\lambda(E_u) - \alpha(u))}{\lambda(E_u \cap^c A)}\right)\right) = \max(0, a(u)\lambda(E_u) - \alpha(u)).$$

Therefore we get $\lambda(\Phi(a, E_\bullet, A)(u)) = \max(0, a(u)\lambda(E_u) - \alpha(u)) + \min(a(u)\lambda(E_u), \alpha(u)) = a(u)\lambda(E_u)$ for all $u \in I$. Finally, since $A \cup E_0$ and $\alpha(0) = \lambda(E_0 \cap A) = \lambda(A) = \lambda(E_0)a(0)$, we get that $\Phi_+(a, E_\bullet, A)(0) = \mathbf{g}(E_0 \cap A, 0) \cup \mathbf{g}(E_0 \cap^c A, 1) = A \cup \emptyset = A$, and this proves the claim. \square

4. PROBABILITY LAW

4.1. The law maps. Recall from [5] that the weak (or coherent) topology on $|\mathcal{K}|$ is the topology such that U is open in $|\mathcal{K}|$ iff $U \cap |F|$ is open for every $F \in \mathcal{K}$, where $|F| = \{\alpha : F \rightarrow [0, 1] \mid \sum_{s \in F} \alpha(s) = 1\}$ is given the topology induced from $[0, 1]^F$. For each $p \geq 1$, we can put a metric topology on the same set, to define a metric space $|\mathcal{K}|_{d_p}$ by the metric $d_p(\alpha, \beta) = \sqrt[p]{\sum_{s \in S} |\alpha(s) - \beta(s)|^p}$. The map $|\mathcal{K}| \rightarrow |\mathcal{K}|_{d_p}$ is continuous, and it is an homeomorphism iff $|\mathcal{K}|$ is metrizable iff it satisfies the first axiom of countability, iff \mathcal{K} is locally finite (see [5] p. 119 ch. 3 sec. 2 theorem 8 for the case $p = 2$, but the proof works for $p \neq 2$ as well).

For $\alpha : S \rightarrow [0, 1]$, we denote $\text{supp}(\alpha) = \{s \in S \mid \alpha(s) \neq 0\}$ the *support* of α . We denote $\Psi_0 : L(\Omega, \mathcal{K}) \rightarrow |\mathcal{K}|$ be defined by associating to a random variable $f \in L(\Omega, \mathcal{K})$ its probability law.

We first prove that Ψ_0 is *not* continuous in general, by providing an example. Let us consider $S = \mathbf{N} = \mathbf{Z}_{\geq 0}$, and $\mathcal{K} = \mathcal{P}_f^*(\mathbf{N})$. We introduce

$$U = \{\alpha \in |\mathcal{K}| \mid \forall s \neq 0 \mid \alpha(s) < 1/\#\text{supp}(\alpha)\}.$$

We note that U is open in $|\mathcal{K}|$. Indeed, if $F \in \mathcal{K}$ we have $U \cap |F| = \{\alpha : F \rightarrow [0, 1] \mid \sum_{s \in F} \alpha(s) = 1 \text{ \& } \forall s \neq 0 \mid \alpha(s) < 1/\#\text{supp}(\alpha)\} = \bigcup_{G \subset F \setminus \{0\}} \{\alpha : G \rightarrow [0, 1] \mid \alpha(0) + \sum_{s \in G} \alpha(s) = 1 \text{ \& } \forall s \in G \mid 0 < \alpha(s) < 1/(\#G + 1)\}$ and it is open as the union of a finite collection of open sets. Now consider $\Psi_0^{-1}(U)$, and let $f_0 \in L(\Omega, \mathcal{K})$ be the constant map $t \mapsto 0$. Clearly $\alpha_0 = \Psi_0(f_0)$ is the map $0 \mapsto 1$, $k \mapsto 0$ for $k \geq 1$, and $\alpha_0 \in U$. If $\Psi_0^{-1}(U)$ is open, there exists $\varepsilon > 0$ such that it contains the open ball centered at f_0 with radius ε . Let n such that $1/n < \varepsilon/3$, and define $f \in L([0, 1], \mathcal{K})$ by $f(t) = 0$ for $t \in [0, 1 - 2/n[$, $f(t) = k$ for $t \in [1 - \frac{2}{n} + \frac{k-1}{2n^2}, 1 - \frac{2}{n} + \frac{k}{(2n-1)n}[$ and $1 \leq k < 2n$, and finally $f(t) = 2n$ for $t \in [1 - \frac{1}{n}, 1]$. We have $d(f, f_0) \leq 2/n < 2\varepsilon/3 < \varepsilon$ hence we should have $\alpha = \Psi_0(f) \in U$. But the support of α has cardinality $2n + 1$, and $\alpha(2n) = 1/n > 1/(2n + 1)$, contradicting $\alpha \in U$.

For short, we denote $|\mathcal{K}|_p = |\mathcal{K}|_{d_p}$. We consider the same ‘law’ map $\Psi : L(\Omega, \mathcal{K}) \rightarrow |\mathcal{K}|_1$. We prove that it is uniformly continuous (and actually 2-Lipschitz). Indeed, if $f, g \in L(\Omega, \mathcal{K})$, and $\alpha = \Psi(f)$, $\beta = \Psi(g)$, then

$$d_1(\alpha, \beta) = \sum_{s \in S} |\alpha(s) - \beta(s)| = \sum_{s \in S} |\lambda(f^{-1}(s)) - \lambda(g^{-1}(s))|$$

and $|\lambda(f^{-1}(s)) - \lambda(g^{-1}(s))| \leq \lambda(f^{-1}(s)\Delta g^{-1}(s))$. But $f^{-1}(s)\Delta g^{-1}(s) = \{t \in f^{-1}(s) \mid f(t) \neq g(t)\} \cup \{t \in g^{-1}(s) \mid f(t) \neq g(t)\}$ whence

$$d_1(\alpha, \beta) \leq \sum_{s \in S} \int_{f^{-1}(s)} d(f(t), g(t)) dt + \sum_{s \in S} \int_{g^{-1}(s)} d(f(t), g(t)) dt = 2 \int_{\Omega} d(f(t), g(t)) dt$$

whence $d_1(\alpha, \beta) \leq 2d(f, g)$. It follows that it induces a continuous map $\bar{L}(\Omega, \mathcal{K}) \rightarrow \overline{|\mathcal{K}|_1}$, where $\overline{|\mathcal{K}|_1} = \{\alpha : S \rightarrow [0, 1] \mid \mathcal{P}_f^*(\text{supp}(\alpha)) \subset \mathcal{K} \text{ \& } \sum_{s \in S} \alpha(s) = 1\}$ endowed with the metric $d(\alpha, \beta) = \sum_{s \in S} |\alpha(s) - \beta(s)|$ is the completion of $|\mathcal{K}|_1$. This map associates to $f \in \bar{L}(\Omega, \mathcal{K})$ the map $\alpha(s) = \lambda(f^{-1}(s))$.

The fact that $|\mathcal{K}|_1$ has the same homotopy type than $|\mathcal{K}|$ has originally been proved by Dowker in [1] in a more general context, and another proof was subsequently provided by Milnor in [4].

It is clear that every mass distribution on the discrete set S is realizable by some random variable. We first show that it is possible to do this *continuously*. In topological terms, this proves the following statement.

Proposition 4.1. *The maps Ψ and $\bar{\Psi}$ admit global (continuous) sections.*

Proof. We define $\sigma : \overline{|\mathcal{K}|_1} \rightarrow \bar{L}(\Omega, \mathcal{K})$ as follows. For any $\alpha \in \overline{|\mathcal{K}|_1}$, $S_\alpha = \text{supp}(\alpha) \subset S$ is countable. Let $A_\pm : S \rightarrow \mathbf{R}_+$ denote the associated cumulative mass functions $A_+(s) = \sum_{u \leq s} \alpha(u)$ and $A_-(s) = \sum_{u < s} \alpha(u)$. They induce increasing injections $(S_\alpha, \leq) \rightarrow [0, 1]$. The map $\sigma(\alpha)$ is defined by $\sigma(\alpha)(t) = a$ if $A_-(a) \leq t < A_+(a)$. Clearly $\bar{\Psi} \circ \sigma$ is the identity. We prove now that σ is continuous. Let $\varepsilon > 0$. There exists $S_\alpha^0 \subset S_\alpha$ finite (and non-empty) such that $\sum_{s \in S_\alpha \setminus S_\alpha^0} \alpha(s) \leq \varepsilon/3$. Let $n = |S_\alpha^0| > 0$. We set $\eta = \varepsilon/3n$. Let $\beta \in \overline{|\mathcal{K}|_1}$ with $|\alpha - \beta|_1 \leq \eta$, and set $B_+(s) = \sum_{u \leq s} \beta(u)$ and $B_-(s) = \sum_{u < s} \beta(u)$. We have

$$d(\sigma(\alpha), \sigma(\beta)) \leq \varepsilon/3 + \sum_{a \in S_\alpha^0} \int_{A_-(a)}^{A_+(a)} d(\sigma(\alpha)(t), \sigma(\beta)(t)) dt$$

Now note that $|A_\pm(a) - B_\pm(a)| \leq |\alpha - \beta|_1 \leq \varepsilon/3n$ for each $a \in S_\alpha^0$ hence

$$\int_{A_-(a)}^{A_+(a)} d(\sigma(\alpha)(t), \sigma(\beta)(t)) dt \leq \frac{2\varepsilon}{3n} + \int_{\max(A_-(a), B_-(a))}^{\min(A_+(a), B_+(a))} d(\sigma(\alpha)(t), \sigma(\beta)(t)) dt = \frac{2\varepsilon}{3n}$$

since $\sigma(\alpha)(t) = \sigma(\beta)(t)$ for each $t \in [\max(A_-(a), B_-(a)), \min(A_+(a), B_+(a))]$, and this yields $d(\sigma(\alpha), \sigma(\beta)) \leq \varepsilon$. Therefore σ provides a continuous global section of $\bar{\Psi}$, which obviously restricts to a continuous global section of Ψ . \square

4.2. Homotopy lifting properties. Let $\Psi_{\mathcal{K}} : L(\Omega, \mathcal{K}) \rightarrow |\mathcal{K}|_1$ and $\bar{\Psi}_{\mathcal{K}} : \bar{L}(\Omega, \mathcal{K}) \rightarrow \overline{|\mathcal{K}|_1}$ denote the law map. If α is a cardinal, we let Ψ_α (resp. $\bar{\Psi}_\alpha$) the map associated to the simplicial complex $\mathcal{P}_f^*(\alpha)$. Recall that a continuous map $p : E \rightarrow B$ is said to have the homotopy lifting property (HLP) with respect to some topological space X if, for any (continuous) maps $H : X \times [0, 1] \rightarrow B$ and $h : X \rightarrow E$ such that $p \circ h = H(\bullet, 0)$, there exists a map $\tilde{H} : X \times [0, 1] \rightarrow E$ such that $p \circ \tilde{H} = H$. A Hurewicz fibration is a map having the HLP w.r.t. arbitrary topological spaces. A Serre fibration is a map having the HLP w.r.t. to every n -spheres, and this is equivalent to having the HLP w.r.t. to any CW-complex.

Lemma 4.2. *If Ψ_α (resp. $\bar{\Psi}_\alpha$) has the HLP wrt the space X , then the map $\Psi_{\mathcal{K}}$ (resp. $\bar{\Psi}_{\mathcal{K}}$) has the HLP wrt the space X for every simplicial complex whose vertex set has cardinality α .*

Proof. This is a straightforward consequence of the fact that the following natural square diagram are cartesian

$$\begin{array}{ccc} L(\Omega, \mathcal{K}) & \hookrightarrow & L_f(\Omega, S) \\ \downarrow & & \downarrow \\ |\mathcal{K}|_1 & \hookrightarrow & |\mathcal{P}_f^*(S)|_1 \end{array} \quad \begin{array}{ccc} \bar{L}(\Omega, \mathcal{K}) & \hookrightarrow & L(\Omega, S) \\ \downarrow & & \downarrow \\ \overline{|\mathcal{K}|_1} & \hookrightarrow & \overline{|\mathcal{P}_f^*(S)|_1} \end{array}$$

\square

Notice that the following lemma applies in particular to every compact metrizable space (e.g. the n -spheres).

Lemma 4.3. *Let X be a separable space. If Ψ_{\aleph_0} (resp. $\bar{\Psi}_{\aleph_0}$) has the HLP wrt the space X then, for every infinite cardinal γ , the map Ψ_γ (resp. $\bar{\Psi}_\gamma$) has the HLP wrt the space X .*

Proof. Let S be a set of cardinality γ , $H : X \times [0, 1] \rightarrow |\mathcal{P}_f^*(S)|_1$ (resp. $\bar{H} : X \times [0, 1] \rightarrow \overline{|\mathcal{P}_f^*(S)|_1}$) and $h : X \rightarrow L_f(\Omega, S)$ (resp. $\bar{h} : X \rightarrow L(\Omega, S)$) be continuous maps such that $\Psi_S \circ h = H(\bullet, 0)$ (resp. $\bar{\Psi}_S \circ \bar{h} = \bar{H}(\bullet, 0)$). Since X is separable, $X \times [0, 1]$ is also separable and so are $H(X \times [0, 1])$ and $\bar{H}(X \times [0, 1])$. Let $(x_n)_{n \in \mathbf{N}}$ be a dense sequence of elements of $H(X \times [0, 1])$

(resp. $\bar{H}(X \times [0, 1])$). Each $\text{supp}(x_n) \subset S$ is countable, and therefore so is $D = \bigcup_n \text{supp}(x_n)$. Let $\alpha \in H(X \times [0, 1])$ (resp. $\alpha \in \bar{H}(X \times [0, 1])$). If $\alpha(s_0) \neq 0$ for some $s_0 \notin D$, there exists x_n such that $d(x_n, \alpha) < \alpha(s_0)$. But since $d(x_n, \alpha) = \sum_{s \in S} |\alpha(s) - x_n(s)|$, this condition implies $x_n(s_0) \neq 0$, contradicting $\text{supp}(x_n) \subset D$. Therefore $\text{supp}(\alpha) \subset D$ for all $\alpha \in H(X \times [0, 1])$ (resp. $\alpha \in \bar{H}(X \times [0, 1])$), and H (resp. \bar{H}) factorizes through a map $H_D : X \times [0, 1] \rightarrow |\mathcal{P}_f^*(D)|_1$ (resp. $\bar{H}_D : X \times [0, 1] \rightarrow |\overline{\mathcal{P}_f^*(D)}|_1$). Note that this implies that h (resp. \bar{h}) takes values in $L_f(\Omega, D)$ (resp. $L(\Omega, D)$), too. By assumption, there exists $\tilde{H}_D : X \times [0, 1] \rightarrow L_f(\Omega, D)$ (resp. $\tilde{\bar{H}}_D : X \times [0, 1] \rightarrow L(\Omega, D)$) such that $\Psi_D \circ \tilde{H}_D = H_D$ and with $\tilde{H}_D(\bullet, 0) = h$ (respectively, \dots). Composing \tilde{H}_D (resp. $\tilde{\bar{H}}_D$) with the natural injection $L_f(\Omega, D) \hookrightarrow L_f(\Omega, S)$ (resp. $L(\Omega, D) \hookrightarrow L(\Omega, S)$) we get the lifting \tilde{H} (resp. $\tilde{\bar{H}}$) we want, and this proves the claim. \square

$$\begin{array}{ccc}
& L_f(\Omega, D) \hookrightarrow L_f(\Omega, S) & \\
\tilde{H}_D \nearrow & \downarrow \Psi_D & \downarrow \Psi_S \\
X \times [0, 1] & \xrightarrow{H} & |\mathcal{P}_f^*(S)|_1 \\
\tilde{H}_D \searrow & \downarrow & \nearrow \\
& |\mathcal{P}_f^*(D)|_1 &
\end{array}$$

Proposition 4.4. *Let X be a topological space and γ a countable cardinal. Then Ψ_γ has the HLP property w.r.t. X as soon as γ is finite or X is compact. Moreover $\bar{\Psi}_\gamma$ has the HLP w.r.t. X as soon as X is compact.*

Proof. Let X be an arbitrary topological space. Our cardinal γ is the cardinal of some initial segment $S \subset \mathbf{N} = \mathbf{Z}_{\geq 0}$, that is $S = [0, m]$ for some m , or $S = \mathbf{N}$. Let $H : X \times [0, 1] \rightarrow |\mathcal{P}_f^*(S)|_1$ and $h : X \rightarrow L_f(\Omega, S)$ such that $H(\bullet, 0) = \Psi_S \circ oh$. For $(x, u) \in X \times [0, 1]$, the element $H(x, u) \in |\mathcal{P}_f^*(S)|_1$ is of the form $(H(x, u)_s)_{s \in S}$, with $\sum_{s \in S} H(x, u)_s = 1$. Therefore, for each $s \in S$, the map $(x, u) \mapsto H(x, u)_s$ defines a continuous map $X \times [0, 1] \rightarrow [0, 1]$.

Let us choose $x \in X$. We construct recursively, for each $n \in \mathbf{N}$, a continuous map $a_n(x, \bullet) \in C^0(I, I)$, maps $\Omega_n(x, \bullet) : I \rightarrow L(2)$, $E_{x, \bullet}^{(n)} : I \rightarrow L(2)$, by letting $E_{x, u}^{(n)} = \Omega \setminus \bigcup_{k < n} \Omega_k(x, u)$, $a_n(x, u) = H(x, u)_n / (1 - \sum_{k < n} H(x, u)_k)$ (with the convention $0/0 = 0$), $A_n(x) = h(x)^{-1}(\{n\}) \in L(2)$, and $\Omega_n(x, u) = \Phi(a_n(x, \bullet), E_{x, \bullet}^{(n)}, A_n(x))(u)$, where Φ is the map afforded by proposition 3.6. From these definitions one gets by induction that $a_n(x, u) \lambda(E_{x, u}^{(n)}) = H(x, u)_n = \lambda(\Omega_n(x, u))$.

The map $\tilde{H} : X \times I \rightarrow L_f(S)$ is then constructed by setting $\tilde{H}(x, u)(t) = n$ if $t \in \Omega_n(x, u)$. Clearly $(\Psi_S \circ \tilde{H}(x, u))_n = \lambda(\Omega_n(x, u)) = H(x, u)_n$ for all n , hence $\Psi_S \circ \tilde{H} = H$. Moreover $\tilde{H}(x, 0)_n = \Omega_n(x, 0) = A_n(x) = h(x)^{-1}(\{n\})$ hence $\tilde{H}(x, 0) = h(x)$ for all $x \in X$.

Therefore it only remains to prove that $\tilde{H} : X \times I \rightarrow L_f(\Omega, S)$ is continuous.

Let us define $\tilde{H}_n : X \times I \rightarrow L(\Omega, \{0, \dots, n\})$ by $\tilde{H}_n(x, u)(t) = \tilde{H}(x, u)(t)$ if $\tilde{H}(x, u)(t) < n$, and $\tilde{H}_n(x, u)(t) = n$ if $\tilde{H}(x, u)(t) \geq n$ (that is, $\tilde{H}_n(x, u)(t) = \min(n, \tilde{H}(x, u)(t))$). For all n , and x, u , we have

$$d(\tilde{H}_n(x, u), \tilde{H}(x, u)) = \lambda \left(\sum_{k > n} \Omega_k(x, u) \right) = \sum_{k > n} H(x, u)_k$$

We first prove that each \tilde{H}_n is continuous. Let $(x_0, u_0), (x, u) \in X \times I$. We have

$$d(\tilde{H}_n(x, u), \tilde{H}_n(x_0, u_0)) = \sum_{k=0}^n \int_{\Omega_n(x_0, u_0)} d(\tilde{H}_n(x, u)(t), \tilde{H}_n(x_0, u_0)(t)) dt$$

hence

$$d(\tilde{H}_n(x, u), \tilde{H}_n(x_0, u_0)) \leq \sum_{k=0}^n \lambda(\Omega_n(x_0, u_0) \Delta \Omega_n(x, u))$$

and therefore it remains to prove that the maps $(x, u) \mapsto \Omega_n(x, u)$ are continuous for each $n \in \mathbf{N}$. We want to prove that $\Omega_n(\bullet, \bullet) \in C^0(X \times [0, 1], L(2))$, which we identify with $C^0(X, C^0([0, 1], L(2)))$ since $[0, 1]$ is (locally) compact. Recall that Φ is continuous $C^0(I, [0, 1]) \times C^0(I, L(2)) \times L(2) \rightarrow C^0(I, L(2))$. Moreover, for arbitrary spaces X, Y, Z and a map $g \in C^0(Y, Z)$, the induced map $C^0(X, Y) \rightarrow C^0(X, Z)$ given by $f \mapsto g \circ f$ is continuous. Letting $Y = C^0(I, [0, 1]) \times C^0(I, L(2)) \times L(2)$ and $Z = C^0(I, L(2))$, we deduce from Φ a continuous map

$$\begin{array}{ccc} \hat{\Phi} : & C^0(X, C^0(I, [0, 1]) \times C^0(I, L(2))) \times L(2) & \longrightarrow C^0(X, C^0(I, L(2))) \\ & \parallel & \parallel \\ & C^0(X \times I, [0, 1]) \times C^0(X \times I, L(2)) \times C^0(X, L(2)) & C^0(X \times I, L(2)) \end{array}$$

By induction and because the maps a_n, A_n are clearly continuous for any n , we get that all the maps involved are continuous, through the recursive identities

- $\Omega_n = \hat{\Phi}(a_n, E_{\bullet, \bullet}^{(n)}, A_n(\bullet))$.
- $E_{x, u}^{(n)} = \Omega \setminus \bigcup_{k < n} \Omega_k(x, u)$

and this proves the continuity of \tilde{H}_n .

If S is finite this proves that \tilde{H} is continuous, because $\tilde{H} = \tilde{H}_n$ for n large enough in this case. Let us now assume $S = \mathbf{N}$ and X is compact. We want to prove that the sequence \tilde{H}_n converges uniformly to \tilde{H} . Since each \tilde{H}_n is continuous this will prove that \tilde{H} is continuous. Let $\varepsilon > 0$. Let $U_n = \{(x, u) \in X \times I \mid \sum_{k \leq n} H(x, u)_k > 1 - \varepsilon\}$. Since H is continuous this defines a collection of open subsets in the compact space $X \times I$, and since $\sum_{k \leq n} H(x, u) \rightarrow 1$ when $n \rightarrow \infty$ for any $(x, u) \in X \times I$, this collection is an open covering. By compactness, and because this collection is a filtration, we have $X \times I = U_{n_0}$ for some $n_0 \in \mathbf{N}$. But then $d(\tilde{H}(x, u), \tilde{H}_n(x, u)) \leq \varepsilon$ for all $n \geq n_0$ and $(x, u) \in U_{n_0} = X \times I$, and this proves the claim. \square

Since it is particularly simpler in this case, we provide an alternative proof for the case of binary random variables.

Corollary 4.5. *The map $\Psi_{\{0,1\}}$ is a Hurewicz fibration.*

Proof. (alternative proof) Let X be a space, and $H : X \times [0, 1] \rightarrow |\mathcal{P}_f^*(2)|_1$ and $h : X \rightarrow L(\Omega, 2)$ such that $H(\bullet, 0) = \Psi_2 \circ h$. Note that $|\mathcal{P}_f^*(2)|_1 = \{\alpha : \{0, 1\} \rightarrow \mathbf{R}_+ \mid \alpha(0) + \alpha(1) = 1\}$ is isometric to $[0, 1]$ through $j : \alpha \mapsto \alpha(1)$, where the metric on $[0, 1]$ is the euclidean one. If $\alpha = \Psi_2 \circ h(x)$, we have $\alpha(0) = 1 - \lambda(h(x))$, $j(\Psi_2(h(x))) = \alpha(1) = \lambda(h(x))$.

Note that $\lambda({}^c \mathbf{g}({}^c A, u)) = u + (1 - u)\lambda(A) = u\lambda(\Omega) + (1 - u)\lambda(A)$. Using the map \mathbf{g} of lemma 3.1 we define $\tilde{\mathbf{g}}(A, a) = \mathbf{g}(A, 1 - a/\lambda(A))$ if $a < \lambda(A)$, and $\tilde{\mathbf{g}}(A, a) = {}^c \mathbf{g}({}^c A, (a - \lambda(A))/(1 - \lambda(A)))$ if $a > \lambda(A)$. It is easily checked that this map extends continuously to $\tilde{\mathbf{g}} : L(2) \times [0, 1] \rightarrow L(2)$ in such a way that $\tilde{\mathbf{g}}(A, \lambda(A)) = A$ for all $A \in L(2)$, and that we have $\lambda(\tilde{\mathbf{g}}(A, a)) = a$ for all A, a .

We define $\tilde{H} : X \times [0, 1] \rightarrow L(\Omega, 2)$ by $\tilde{H}(x, u) = \tilde{\mathbf{g}}(h(x), j(H(x, u)))$. We have $\lambda(\tilde{H}(x, u)) = j(H(x, u))$ hence $\Psi_2 \circ \tilde{H} = H$, and $\tilde{H}(x, 0) = h(x)$ for all $x \in X$. This proves the claim. \square

Altogether, these statements imply the following result, which completes the proof of theorem 1.1.

Theorem 4.6. *For an arbitrary simplicial complex, the maps $\Psi_{\mathcal{K}}$ and $\overline{\Psi}_{\mathcal{K}}$ are Serre fibrations and weak homotopy equivalences. If \mathcal{K} is finite, then $\Psi_{\mathcal{K}}$ and $\overline{\Psi}_{\mathcal{K}}$ are Hurewicz fibrations and homotopy equivalences.*

Proof. Let \mathcal{K} be an arbitrary simplicial complex. We first prove that $\Psi_{\mathcal{K}}$ and $\overline{\Psi}_{\mathcal{K}}$ are Serre fibrations. By lemmas 4.2 and 4.3, and since the n -spheres are separable spaces, we can restrict ourselves to Ψ_{γ} and $\overline{\Psi}_{\gamma}$ for $\gamma \leq \aleph_0$, and this is true in this case because the n -spheres are compact, by proposition 4.4. Let us choose $\{x_0\} \in \mathcal{K}$, and $\tilde{x}_0 : S \rightarrow [0, 1]$ being given by $x_0 \mapsto 1$ and $x \mapsto 0$ if $x \neq x_0$. Then $\tilde{x}_0 \in |\mathcal{K}|_1$, and the fiber above it $\Psi_{\mathcal{K}}^{-1}(\{\tilde{x}_0\}) = \Psi_{\mathcal{K}}^{-1}(\{\tilde{x}_0\})$ is a point. Since these maps are Serre fibrations this implies that they are weak homotopy equivalences. If \mathcal{K} is a

finite simplicial complex, by lemma 4.2 and proposition 4.4 we get that $\Psi_{\mathcal{K}}$ and $\overline{\Psi}_{\mathcal{K}}$ are Hurewicz fibrations, and picking again some $\{x_0\} \in \mathcal{K}$ we get that the fiber above some point is itself a point, whence the homotopy fiber of these Hurewicz fibrations is contractible and they are homotopy equivalences. This proves the claim. □

REFERENCES

- [1] C.H. Dowker, *Topology of Metric Complexes*, American J. Math. **74** (1952), 555-577.
- [2] M. Keane, *Contractibility of the automorphism group of a nonatomic measure space*, Proc. Amer. Math. Soc. **26** (1970), 420-422.
- [3] I. Marin, *Classifying spaces and measure theory*, preprint 2017, arXiv:1702.01889v1.
- [4] J. Milnor, *On Spaces having the homotopy type of a CW-complex*, Trans. A.M.S. **90** (1959) 272-280.
- [5] E.H. Spanier, *Algebraic Topology*, Springer-Verlag, 1966.

LAMFA, UMR CNRS 7352, UNIVERSITÉ DE PICARDIE-JULES VERNE, AMIENS, FRANCE
E-mail address: `ivan.marin@u-picardie.fr`